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Stochastic resonance in second-order autonomous systems subjected only to white noise

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Abstract
In this article, interwell stochastic resonance (SR) as well as intrawell SR in an under-damped stochastic system without periodic modulation is presented according to different types of stable equilibrium point on the one-dimensional global attractor. The corresponding mechanism is convincingly explained based on the discussion of the attracting ability of the global attractor.

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1. Introduction

Since its introduction in the 1980s by Benzi *et al* [1] and Nicolis *et al* [2], stochastic resonance (SR) has aroused continuous interest (for a recent review, see [3]). Originally, SR referred to a cooperative phenomenon of a weak periodic force and noise in a nonlinear system, where the output of the system (usually measured by the signal-to-noise ratio or the response amplitude) undergoes a resonance-like behaviour as a function of noise. The physical literature is rife with studies on this aspect [4–11].

The extended type of SR is without a periodic driving force. This means that under a constant drifting force, pure noise can also induce coherent motion, and with increasing noise intensity the height of the spectrum peak as well as the suitably defined quality factor goes through a bell-shaped maximum [12–16]. Different to the conventional SR in which a periodic force is presented, this phenomenon of SR reflects the intrinsic periodicity of the system under a constant drifting term and was further observed in some biochemistry reactions [17].

In this paper, we explore the later phenomenon of SR of an under-damped single pendulum driven by a constant force plus noise perturbation. The corresponding stochastic differential equation in dimensionless form reads

$$\ddot{\theta} + \alpha\dot{\theta} + \sin\theta = b + D\xi(t) \tag{1}$$

where $b > 0$, $\alpha > 0$ is the damping coefficient and $\xi(t)$ is a Gaussian white noise satisfying $\langle \xi(t) \rangle = 0$, $\langle \xi(t)\xi(t') \rangle = \delta(t - t')$. This equation appears in different contexts, for example the case of a Josephson junction [18, 19], charge density waves [20], motion of fluxons in superconductors [21] etc. Therefore, it is of practical significance to beat out the SR parameter regime.

The main purpose of this paper is to give a systematic investigation of SR in system (1). To do this, in section 2, we first analyse the deterministic dynamical behaviour of equation (1) for various values of parameters, where the global attractor is mathematically manifested and the b–α parameter space is compartmentalized into three regions according to the different types of global attractor. Then in section 3, we investigate the influence of the noise in these three parameter regions. We show that when the global attractor is formed by a stable node, a saddle point and the heteroclinic orbits between them, interwell SR exists even in weakly damped cases. Interestingly, we find that when the stable node on the attractor curve is replaced by a stable focus, no interwell SR occurs because of the occurrence of cycle skipping phenomena, which means the switching motion can hardly show any periodicity. However, the added noise can still induce coherent rotation around the focal point, which results in the occurrence of intrawell SR. As the global attractor includes a stable limit cycle, the role of the noise can either be destructive or constructive.

2. The deterministic dynamical behaviour

Numerical results of the dynamic of equation (1) in the absence of noise have been treated in [22] and the references therein. What we do in the following is to further mathematically demonstrate the global attractor of the deterministic system by applying the idea in [23] as well the results in [24]. Though simple, it is crucial for clarifying the mechanism of SR in section 3.

After being written as a set of first-order equations, the deterministic system can also be characterized as

$$\begin{aligned} \dot{\theta} &= \phi \\ \dot{\phi} &= -\alpha\phi + b - \sin\theta. \end{aligned} \tag{2}$$

The phase space of equation (2) can either be regarded as the plane R^2 or the cylinder $E^2 = S^1 \times R^1$ because of the periodicity of θ. Obviously, equation (2) has two equilibrium points for $0 < b < 1$, one fixed point for $b = 1$ and no equilibrium point for $b > 1$. As for limit cycles, first we have:

Proposition. *System (2) has no limit cycle of type I on the cylinder E^2.*

Proof. Suppose that D is any bounded region on R^2 and P^t is the dynamical flow of equation (2), then

$$\int\int_{P^t D} \mathrm{d}\theta\, \mathrm{d}\phi = \int\int_D |\det M(t)|\, \mathrm{d}\theta\, \mathrm{d}\phi$$

where $M(t)$ is the Jacobian; with a little computation, one has

$$\frac{\mathrm{d}}{\mathrm{d}t} M(t) = \begin{bmatrix} 0 & 1 \\ -\cos\theta(t) & -\alpha \end{bmatrix} M(t) \qquad M(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Hence

$$\frac{\mathrm{d}}{\mathrm{d}t}(\det(M(t))) = -\alpha(\det(M(t))).$$

As a result, $|\det M(t)| = \mathrm{e}^{-\alpha t}$, and therefore,

$$\int\int_{P^t D} \mathrm{d}\theta\,\mathrm{d}\phi = \mathrm{e}^{-\alpha t}\int\int_D \mathrm{d}\theta\,\mathrm{d}\phi.$$

This means that the area of any bounded region D is contracted after the map P^t and tends to zero if t tends to infinity. Therefore, there is no limit cycle of type I on E^2. □

Concerning the existence of the second type of limit cycle, for $b > 1$, it was proved in [24] that there exists a unique stable limit cycle on E^2. As for $0 < b \leqslant 1$, here we cite the Tricomi theorem from [24] without proof:

Theorem (F Tricomi). *For any fixed value of $b(0 < b \leqslant 1)$, there exists correspondingly a unique value of $\alpha(b)$ such that for any $0 < \alpha \leqslant \alpha(b)$, system (2) has a unique stable limit cycle of type II on the cylinder E^2, while for $\alpha > \alpha(b)$, no limit cycle exists.*

In the following we shall prove that, though no limit cycle exists for $\alpha > \alpha(b)$, there is still a one-dimensional global attractor. The idea comes from [23].

From equation (2), we have

$$\frac{\mathrm{d}\dot\theta}{\mathrm{d}\theta} = -\alpha + \frac{b-\sin\theta}{\dot\theta}. \tag{3}$$

Multiplying equation (3) by $\dot\theta$ and then integrating from $\theta = -\pi$ to $+\pi$ yields

$$\begin{aligned}\tfrac{1}{2}[\dot\theta^2(\pi) - \dot\theta^2(-\pi)] &= 2b\pi - \alpha\int_{-\pi}^{\pi}\dot\theta\,\mathrm{d}\theta\\ &= 2b\pi - \pi\alpha[\dot\theta(\pi) + \dot\theta(-\pi)] + \alpha\int_{-\pi}^{\pi}\theta\left[-\alpha + \frac{b-\sin\theta}{\dot\theta}\right]\mathrm{d}\theta.\end{aligned}$$

If $\dot\theta$ is sufficiently large, the terms in θ in the integrand will not change appreciably during one cycle, so that throughout the cycle $\dot\theta \approx \dot\theta(-\pi)$ for $-\pi \leqslant \theta \leqslant \pi$. Then

$$\begin{aligned}\frac{1}{2}[\dot\theta^2(\pi) - \dot\theta^2(-\pi)] &\approx 2\pi\left[b - \alpha\dot\theta(-\pi) + \frac{\alpha}{\dot\theta(-\pi)}\right]\\ &= -\frac{2\pi}{\dot\theta(-\pi)}[\alpha\dot\theta^2(-\pi) - b\dot\theta(-\pi) - \alpha].\end{aligned}$$

Since $\Delta = b^2+4\alpha^2 > 0$ for all $b > 0, \alpha > 0$, then $\frac{1}{2}[\dot\theta^2(\pi)-\dot\theta^2(-\pi)] < 0$ for sufficiently large value of $\dot\theta$, so any trajectory starting far above the θ-*axis* will decay. Obviously, the curve (denoted as Γ) composed of the equilibrium points and the heteroclinic orbits between them is invariant and compact on E^2 under the map P_t. Since there is no limit cycle for $\alpha > \alpha(b)$, then any orbits above Γ will be attracted by this curve. Otherwise, there will exist at least a semistable limit cycle. This contradicts the above theorem. Similarly, $\frac{1}{2}[\dot\theta^2(\pi)-\dot\theta^2(-\pi)] > 0$ if $\dot\theta$ is sufficiently small, then all the trajectories below Γ will also be attracted to this curve. So Γ is the one-dimensional global attractor of equation (2) for $\alpha > \alpha(b)$.

Taking the different types of stable equilibrium point on Γ into account, the whole 'phase diagram' can then be compartmentalized into the following three parts (see figure 1):

$$\begin{aligned}A &= \left\{(b,\alpha)|\alpha > \alpha(b), \alpha^2 \geqslant 4\sqrt{1-b^2}, 0 < b \leqslant 1\right\}\\ B &= \left\{(b,\alpha)|\alpha > \alpha(b), \alpha^2 < 4\sqrt{1-b^2}, 0 < b < 1\right\}\\ C &= \{(b,\alpha)|0 < \alpha \leqslant \alpha(b), 0 < b \leqslant 1\} \cup \{(b,\alpha)|b > 1\}.\end{aligned}$$

In every part, system (2) has different types of global attractor (see figure 2).

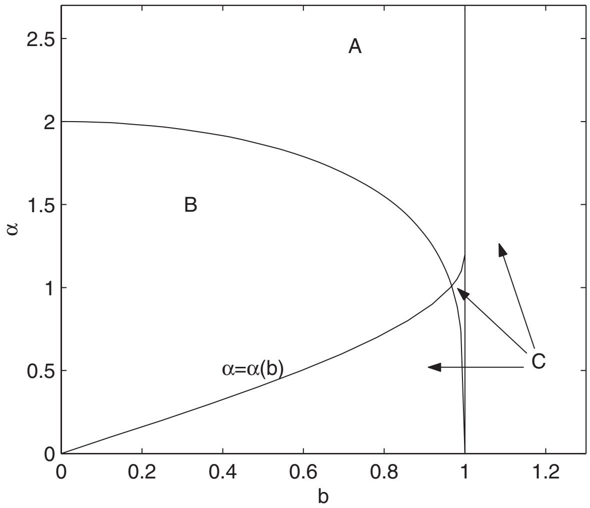

Figure 1. The three regions of the system parameters where equation (2) displays different dynamics.

3. Stochastic resonance without periodic force

Now let us investigate the influence of the noise on system (2) in each of the above three regions. Here power spectra of 500 runs of the time series $\{\sin\theta(t)\}$ are averaged (taking $\{\sin\theta(t)\}$ because the SR considered here is confined on a manifold), and, following [12], we characterize the noise-induced effect by the appropriate quality factor β taken as $\beta = \omega_p h/W$, where h is the maximal peak height of the power spectrum, W is the width of the spectrum measured at the height of $h/\sqrt{e}$ and ω_p is the peak frequency.

Case 1. $(b, \alpha) \in A$. For illustration, we take $\alpha = 2.5$, $b = 0.95$. Without any noise perturbation, the particle will eventually approach the stable node along the curve Γ. When a small amount of noise is included, only random circulations of the particle near the attractor on the cylinder appear. Correspondingly, in figure 3(a) one can only see a small peak at a small frequency in the profile of the power spectrum.

As the noise intensity properly increases, it becomes easier for the particle to escape from the stable node. Meanwhile, the global attractor can still prevent the particle from leaving it too far away for most of the time. Thus with increasing noise intensity D, the switches between the stable identical nodes on R^2 (or circulations on E^2) show a better degree of coherence (see figure 5(a)). Then the height of the spectrum peak as well as the peak frequency ω_p increases (see figure 3(b)). After h reaches a maximum at a certain value of D, it decreases as D further increases. This is because the global attracting ability of the curve Γ is gradually suppressed by the randomness of the noise and the switching events are frequently accompanied by large bursts of the particle away from Γ. If the noise intensity becomes too large, Γ can hardly resist the strong disturbance of the noise and the average motion of the particle exhibits no definite direction. That is to say, the system is completely dominated by the white noise. The spectrum in figure 3(c) confirms this fact.

According to the above investigation, we say that interwell SR happens at a certain noise intensity. This conclusion can be further seen from the bell-shaped curve of the quality factor β versus D in figure 4.

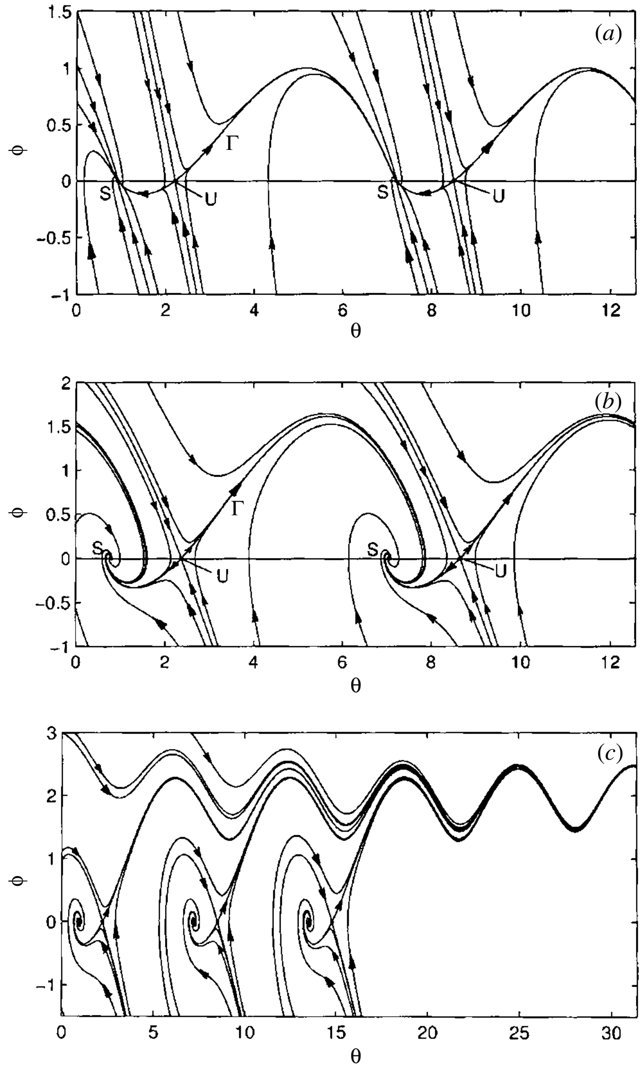

Figure 2. Phase portraits of equation (2) for $(b, \alpha) \in A$ (*a*), $(b, \alpha) \in B$ (*b*) and $(b, \alpha) \in C$ (*c*). The global attractor of system (2) in these three cases can be clearly seen.

Case 2. $(b, \alpha) \in B$. Now let us see the case when the stable node is replaced by a focal point. Here we take $\alpha = 0.97$, $b = 0.95$ and plot the corresponding average power spectra for different values of D in figure 6. Unlike the case of $(b, \alpha) \in A$, it is shown that for small D there exist two spectrum peaks in the profile of the average power spectrum. One is at a very small frequency with height h_1 and another is at a larger frequency with height $h_2 < h_1$ (figure 6(*a*)). When the noise intensity increases, both the peaks increase and the first one is still higher than the second one (figure 6(*b*)). After h_1 reaches its maximum at a certain value of D, it decreases and the first peak gradually becomes vague, while h_2 still increases with increasing D (figure 6(*c*)). Subsequently, only one peak appears in the profile of the power spectrum. Then after another critical value of D, h_2 begins to decrease with the further increase of noise intensity and no obvious peak can be observed for sufficiently large D (figure 6(*d*)).

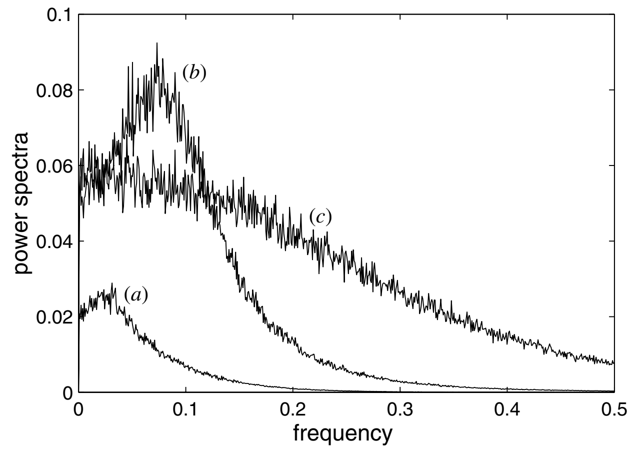

Figure 3. Power spectra of $\{\sin\theta(t)\}$ for the case $(b, \alpha) \in A$ with $\alpha = 2.5, b = 0.95$ and $D = 0.2$ (a), $D = 0.9$ (b) and $D = 3$ (c).

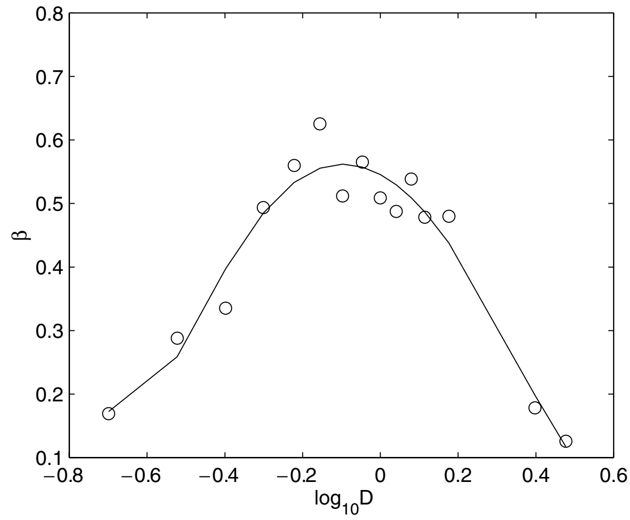

Figure 4. The quality factor β versus $\log_{10}(D)$ for the case $\alpha = 2.5, b = 0.95$.

In order to determine which peak characterizes the interwell motion, we plot the average power spectra for $\alpha = 0.97, b = 0.5, D = 0.2$ and for $\alpha = 2.5, b = 0.5, D = 0.3$ in figure 7(a) and (b), respectively. For the former case the stable fixed point is a focal point and there is an obvious spectrum peak, while for the latter the stable fixed point is a node, but there is no obvious peak in the power spectrum. Numerical simulations show that, in either case, the system can hardly switch between the identical stable fixed points (see figure 7(c) and (d)). This manifests that the obvious peak in figure 3 is really caused by the switches between the stable nodes, while the second peak in figure 6 is actually caused by the circular motion around the stable focal point. So this spectrum peak in figure 6 should not be taken as the measure of the interwell SR.

The question is why the first peak in figure 6 is at such a small frequency. To explore this, we compare the corresponding Lyapunov exponent E_2^{L} of the case $(b, \alpha) \in B$ with that of the case $(b, \alpha) \in A$. In the deterministic case, the Lyapunov exponents reflect the exponential

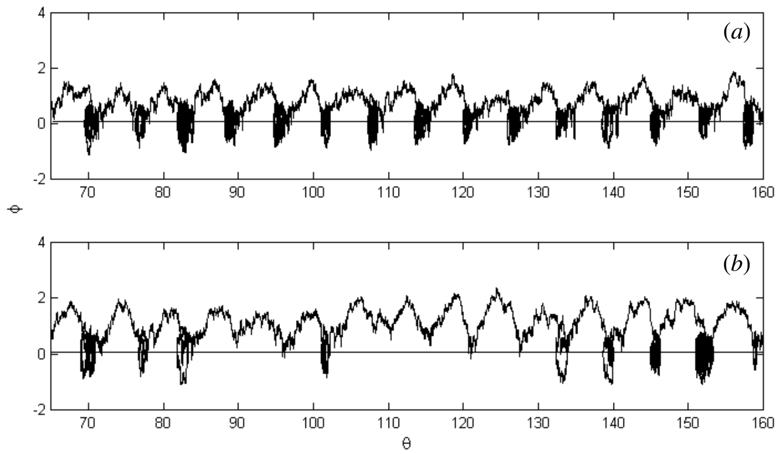

Figure 5. The trajectories (θ, ϕ) of system (1) for (*a*) $(b, \alpha) \in A$ with $\alpha = 2.5$, $b = 0.95$, $D = 0.4$ and (*b*) $(b, \alpha) \in B$ with $\alpha = 0.97$, $b = 0.95$, $D = 0.5$. The cycle skipping phenomena in the case $(b, \alpha) \in B$ can be clearly seen.

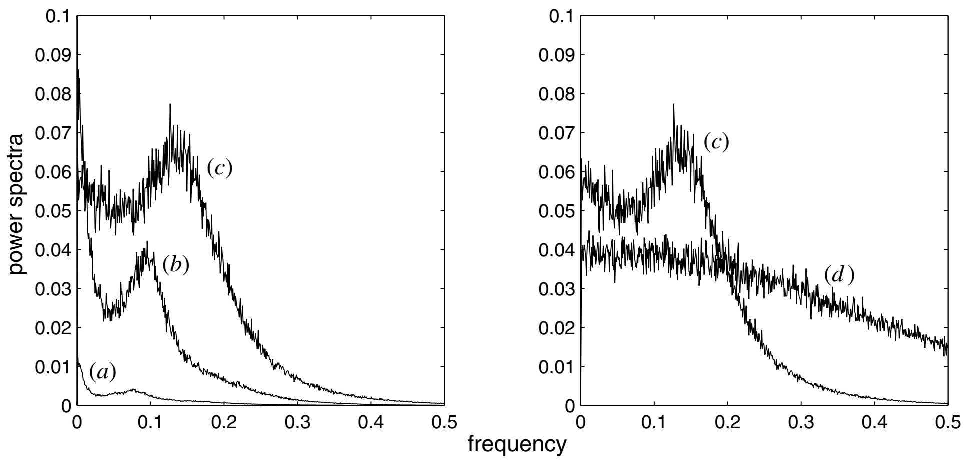

Figure 6. Power spectra of $\{\sin\theta(t)\}$ for the case $(b, \alpha) \in B$ with $\alpha = 0.97, b = 0.95$ and $D = 0.1$ (*a*), $D = 0.3$ (*b*) $D = 0.7$ (*c*) and $D = 3$ (*d*).

Table 1. The Lyapunov exponent E_2^{L} for $b = 0.95$ and for various values of α.

	$\alpha = 0.8$	$\alpha = 0.85$	$\alpha = 0.97$	$\alpha = 1.0$	$\alpha = 1.3$	$\alpha = 1.5$	$\alpha = 2.0$	$\alpha = 2.5$
type of stable fixed point	focus	focus	focus	focus	node	node	node	node
E_2^{L}	−0.4005	−0.4225	−0.4855	−0.5006	−0.9817	−1.2491	−1.8286	−2.3675

approaching rate of a trajectory to the attracting curve Γ. In table 1, the values of E_2^{L} for different values of α with fixed $b = 0.95$ are calculated based on the method in [25].

From table 1, one can see that the absolute values of E_2^{L} for $\alpha = 0.97$ are much smaller than those for $\alpha = 2.5$ in the deterministic case. When a certain amount of noise is included,

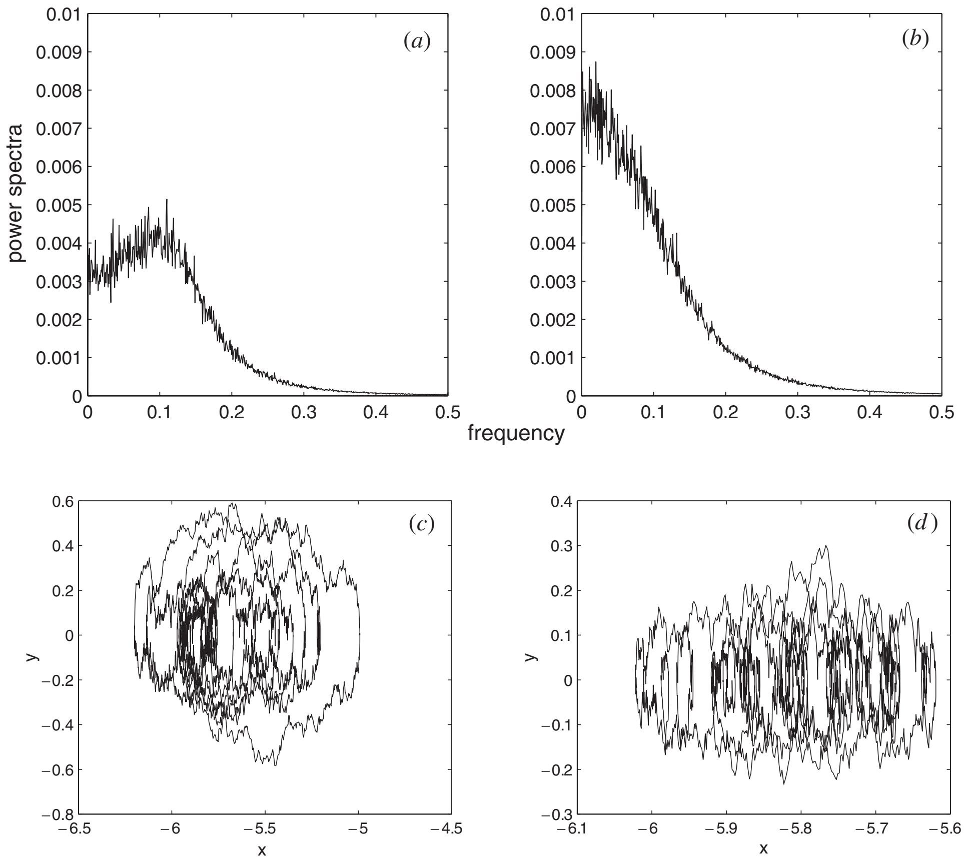

Figure 7. Power spectra of $\{\sin\theta(t)\}$ for the case $(b, \alpha) \in B$ with $\alpha = 0.97$, $b = 0.5$, $D = 0.2$ (*a*) and for the case $(b, \alpha) \in A$ with $\alpha = 2.5$, $b = 0.5$, $D = 0.3$ (*b*). (*c*) and (*d*) are the trajectories corresponding to the parameters in (*a*) and (*b*), respectively. In both cases, the particle cannot escape from the stable fixed point to switch to the next stable fixed point, but there are different profiles of the power spectra.

in the case $(b, \alpha) \in B$, the particle can escape from the stable focal point, but before entering the attracting basin near the focal point again, it may have wound on the cylinder for several cycles, the number of which is random (see figure 5(*b*)). This is called the cycle skipping phenomenon. However, in the case $(b, \alpha) \in A$, because of the strong attracting ability of the stable nodes, such a phenomenon rarely occurs (see figure 5(*a*)). Thus the switches between the stable nodes on R^2 exhibit a periodic nature, and therefore there is a well defined SR peak in the profile of the average power spectrum. But, in the case $(b, \alpha) \in B$, due to the cycle skipping phenomena, the switches between the stable focal points become totally aperiodic, i.e. the periodicity of such events is infinitely long. Consequently, the peak in the average power spectrum only appears at a very small frequency. For this reason, we conclude that interwell SR can hardly happen in the case $(b, \alpha) \in B$.

However, excluding the part at small frequencies, the height of the second spectrum peak as well as the corresponding quality factor still undergoes a process of first increasing, reaching a maximum and then decreasing with the increase of the noise intensity (see figure 8). This means that the circular motion around the focal point shows the best degree of coherence at an optimal noise intensity. In this parameter regime therefore, we conclude that intrawell

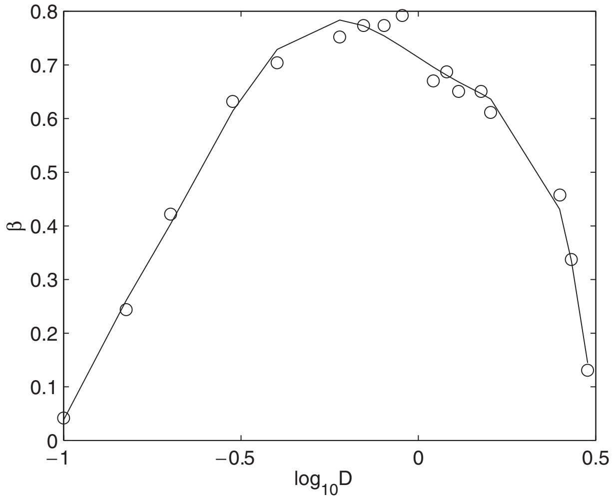

Figure 8. The quality factor β measuring the coherence of the intrawell motion versus $\log_{10}(D)$ for the case $\alpha = 0.97$, $b = 0.95$.

SR exists. The mechanism is due to the circulative way of the particle approaching the stable focal point in the deterministic case. Because of this, there is a favourable direction of rotation around the stable focal point when noise is applied, and with the proper increase of the noise intensity such intrawell rotation shows a better degree of coherence.

Here, we need to point out that if the parameters (b, α) lie near the critical curve of regions A and B, it is hard to determine whether the SR is caused by interwell switches between the stable fixed points or circular motion around the stable fixed point.

Case 3. $(b, \alpha) \in C$. In this case, for $b > 1$, the noise can only play destructive roles because the particle already rotates around the limit cycle in the deterministic case. As for $0 < b \leqslant 1$, the long-time dynamics of the deterministic system is either rotating around the limit cycle or resting near the stable (semi-stable) focal point depending on the selection of the initial value. Then the added noise may either destroy the original periodic motion or activate the particle from its rest state to rotating around the limit cycle. We shall not present them further here.

4. Concluding remarks

We have given a systematic investigation of the existence of SR in the under-damped autonomous system without periodic modulation. We showed that when there is a stable node on the one-dimensional global attractor, a moderate amount of noise not only motivates the system to switch between the identical stable nodes, but also controls the timescale to induce some periodicity of the switching events. Moreover, interwell SR occurs when the circulation near the attracting curve displays the best degree of coherence. When the stable node on the attracting curve is replaced by a stable focal point, the attracting ability of the global attractor is weak; then the switches between the identical stable focal points are accompanied by the occurrence of cycle skipping phenomena, which make the switching events totally aperiodic, so no interwell SR can be observed in this case. However, the added noise can still induce coherent circular motion around the focal point, and intrawell SR exists in this parameter regime.

Acknowledgments

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